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Algebra Structures for General Designs*

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1. INTRODUCTION

In this paper we discuss designs of type $(0, t)$ in the sense of Dembowski [1], i.e., we drop the requirement of constant block size. Then it is possible to define an addition and three multiplications for these general t-designs. The algebra structures arising in this way will be completely determined.

1.1. Let V denote a finite, nonempty set, and let its cardinality, $|V| = v$. Usually we take $V = \{1,..., v\}$. The subsets $x \subseteq V$ will be called blocks, and we associate nonnegative, integral multiplicities $c(x)$ with every block. Such a function c will be called an exact cover if there exists a constant $\lambda_1(c)$ such that for every point $p \in V$

$$\sum_{x \ni p} c(x) = \lambda_1(c).$$

Here λ_1 is the replication number or depth of the cover: it tells how many times each point p is covered by c. Note that the sum of exact covers (as functions) is the exact cover obtained by "superposition".

We remark that exact covers are related to what Shapley [8] calls "balanced sets": a collection of blocks is balanced if there exists an exact cover using exactly these blocks.

1.2. For exact covers we define three multiplications $c_1 * c_2$, $c_1 \wedge c_2$, $c_1 \vee c_2$ based on the Boolean sum (symmetric difference) $x + y$, the Boolean product (intersection) xy, and the union $x \cup y$, respectively, $(x, y \subseteq V)$:

$$(c_1 * c_2)(z) = \sum_{x+y=z} c_1(x)\, c_2(y),$$

$$(c_1 \wedge c_2)(z) = \sum_{xy=z} c_1(x)\, c_2(y),$$

$$(c_1 \vee c_2)(z) = \sum_{x\cup y=z} c_1(x)\, c_2(y).$$

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These may be called sum-convolution, product-convolution, and union-convolution, respectively. We shall see that all of them represent exact covers again.

1.3. The number of blocks in the cover c equals

$$\sum_x c(x) = \lambda_0(c)$$

and will be called the weight of the cover. It is interesting to see how the parameters λ_0, λ_1 change with the operations:

$$\begin{aligned}
&\lambda_0(c_1 + c_2) = \lambda_0(c_1) + \lambda_0(c_2), \qquad && \lambda_1(c_1 + c_2) = \lambda_1(c_1) + \lambda_1(c_2),\\
&\lambda_0(c_1 \wedge c_2) = \lambda_0(c_1)\, \lambda_0(c_2), && \lambda_1(c_1 \wedge c_2) = \lambda_1(c_1)\, \lambda_1(c_2),\\
&\lambda_0(c_1 * c_2) = \lambda_0(c_1)\, \lambda_0(c_2), && \lambda_0(c_1 \vee c_2) = \lambda_0(c_1)\, \lambda_0(c_2),\\
&\lambda_1(c_1 * c_2) = \lambda_0(c_1)\, \lambda_1(c_2) + \lambda_1(c_1)\, \lambda_0(c_2) - 2\lambda_1(c_1)\, \lambda_1(c_2), &&\\
&\lambda_1(c_1 \vee c_2) = \lambda_0(c_1)\, \lambda_1(c_2) + \lambda_1(c_1)\, \lambda_0(c_2) - \lambda_1(c_1)\, \lambda_1(c_2). &&
\end{aligned}$$

1.4. It is natural to allow arbitrary rational multiplicities $c(x)$ in the exact covers c. Then the collection C_1 of all these c's forms a commutative algebra over the rationals Q with operations $(+, *)$ or $(+, \wedge)$ or $(+, \vee)$. Let $X = X(V)$ denote the collection of all subsets of V, and let Q^X denote the vector space (over Q) of all functions from X into Q. Adding the operation $*$, we obtain the algebra $Q_*{}^X$, which is the group algebra over Q of the additive Boolean group X. Similarly, adding the operations $\wedge$ or $\vee$, we obtain the algebras $Q_\wedge{}^X$ and $Q_\vee{}^X$, which are the semigroup algebras over Q of the multiplicative Boolean semigroup X or the semigroup X under union. Then C_1 forms a subalgebra of $Q_*{}^X$, of $Q_\wedge{}^X$, and of $Q_\vee{}^X$.

The semi-group algebra $Q_\wedge{}^X$ has recently been used by Rota [7] in connection with valuations or additive functions $c \in Q^X$ characterized by

$$c(x \cup y) + c(x \cap y) = c(x) + c(y).$$

These valuations are, in a natural way, orthogonal to exact covers with parameters $\lambda_0 = \lambda_1 = 0$, also see Schapley [8].

1.5. It is possible to study balanced incomplete block designs within this framework if we do not require constant block size. Such designs are characterized as exact covers (with nonnegative, integral multiplicities) satisfying

$$\sum_{x \ni p,q} c(x) = \lambda_2(c)$$

for every pair $p \neq q$ in V. We shall see that addition and all of the multiplications produce designs of the same type, in fact,

$$\lambda_2(c_1 + c_2) = \lambda_2(c_1) + \lambda_2(c_2), \qquad \lambda_2(c_1 \wedge c_2) = \lambda_2(c_1)\,\lambda_2(c_2),$$

$$\begin{aligned}\lambda_2(c_1 * c_2) &= 2\lambda_1(c_1)\,\lambda_1(c_2) + (\lambda_0(c_1) - 4\lambda_1(c_1))\,\lambda_2(c_2)\\ &\quad + \lambda_2(c_1)\,(\lambda_0(c_2) - 4\lambda_1(c_2)) + 4\lambda_2(c_1)\,\lambda_2(c_2),\\ \lambda_2(c_1 \vee c_2) &= \lambda_0(c_1)\,\lambda_2(c_2) + \lambda_2(c_1)\,\lambda_0(c_2) + 2\lambda_1(c_1)\,\lambda_1(c_2) + \lambda_2(c_1)\,\lambda_2(c_2)\\ &\quad - 2\lambda_1(c_1)\,\lambda_2(c_2) - 2\lambda_2(c_1)\,\lambda_1(c_2).\end{aligned}$$

If we allow rational multiplicities we obtain a collection C_2 which is a subalgebra of C_1 in all three interpretations.

1.6. More generally, we may consider t-designs if we drop the requirement of constant block size: Let V_t denote the collection of all t-sets in V $(t = 0, 1,..., v)$ and require

$$\sum_{x \supseteq y} c(x) = \lambda_s(c), \qquad y \in V_s\,, \qquad \text{for } s = 0, 1,..., t. \tag{1.1}$$

For general t-designs we admit nonnegative, integral multiplicities. If we admit rational multiplicities we call such a function c a t-exact cover. The collection of 1-exact covers is C_1 from above; the collection of 2-exact covers is C_2 from above; generally we denote the collection of t-exact covers by $C_t = C_t(X)$. In a trivial way $C_0 = \underset{\sim}{Q}^X$, which we might call the collection of "partial covers" of V. We have

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_v\,,$$

and each C_t is an algebra in three ways.

1.7. Our main objective is to develop the algebra structures of $C_t(X)$. Since we deal with convolution algebras it is natural to introduce Fourier transforms (Section 2), see, for example, Hewitt and Zuckerman [5]. The matrices of those Fourier transforms are seen to be members of a special class of matrices which we have called combinatorial matrices. In this general context we discuss the properties and interrelations among the matrices of the Fourier transforms (Section 3). The values of the Fourier transforms are related to the parameters, thereby dispalying the degrees of freedom and the structure of C_t (Section 4). The exact covers ($t = 1$) play a dominating role and can be used in an inductive description of C_t if one replaces X by more general spaces Y (Section 5). Of course, it would be more desirable to restrict the multiplicities to be integers or to be nonnegative integers. We denote the corresponding collections of t-exact covers by $C_t[X]$ and $C_t^+[X]$, respectively.

Our results in that direction are less complete (Section 6). Furthermore, it is desirable to restrict the support of c to smaller sets like V_k, which means that all blocks used in the cover are of size k. Then the algebra structure is lost, but the linear structure can still be determined. This is done in [4].

2. Fourier Transforms

For $c \in Q^X$ we define $\bar{c}$, $\hat{c}$, $\check{c} \in Q^X$ by

$$\bar{c}(x) = \sum_y (-1)^{|xy|} c(y), \qquad y \in X, \tag{2.1}$$

$$\hat{c}(x) = \sum_{y \supseteq x} c(y), \qquad y \in X, \tag{2.2}$$

$$\check{c}(x) = \sum_{y \subseteq x} c(y), \qquad y \in X. \tag{2.3}$$

It is convenient to decide upon some linear order for the blocks X and to think of c as a vector with components $c(x)$. Then we may write

$$\bar{c} = Fc, \qquad \hat{c} = Gc, \qquad \check{c} = Hc,$$

where the matrices F, G, H have the following entries in the (x, y)-position.

$$F_{xy} = \langle x, y \rangle = (-1)^{|xy|};$$
$$G_{xy} = [x, y] = \begin{Bmatrix} 1 \text{ if } x \subseteq y \\ 0 \text{ otherwise} \end{Bmatrix};$$
$$H_{xy} = [y, x].$$

2.1. Since $|y_1 + y_2| \equiv |y_1| + |y_2| \bmod 2$ we have

$$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle \langle x, y_2 \rangle, \qquad \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle \langle x_2, y \rangle, \tag{2.4}$$

and trivially

$$[x, y_1 y_2] = [x, y_1][x, y_2], \qquad [x_1 \cup x_2, y] = [x_1, y][x_2, y]. \tag{2.5}$$

These are the character properties which insure that

$$\bar{c}_1(x)\, \bar{c}_2(x) = \overline{c_1 * c_2}(x), \qquad \hat{c}_1(x)\, \hat{c}_2(x) = \widehat{c_1 \wedge c_2}(x),$$
$$\check{c}_1(x)\, \check{c}_2(x) = \widecheck{c_1 \vee c_2}(x). \tag{2.6}$$

With respect to transposition we observe that

$$F^T = F \qquad \text{and} \qquad G^T = H.$$

We may define the operation of complementation:

$$\tilde{c}(x) = c(V + x).$$

The matrix K of this operation is defined:

$$K_{xy} = \begin{cases} 1 & \text{if } \ x = V + y \\ 0 & \text{otherwise} \end{cases}.$$

Clearly K is a symmetric permutation matrix and obviously we have the identity

$$GK = KH. \tag{2.7}$$

It is then clear that

$$\tilde{c} * \tilde{d} = c * d, \qquad \tilde{c} \wedge \tilde{d} = \widetilde{c \vee d}, \qquad \tilde{c} \vee \tilde{d} = \widetilde{c \wedge d}. \tag{2.8}$$

2.2. The binomial theorem implies

$$\sum_{y \subseteq x} a^{|y|} = (1 + a)^{|x|}, \qquad a \in Q,$$

with the understanding that $0^0 = 1$. Essentially a special case of that is the orthogonality relation

$$\sum_y \langle x, y \rangle = \begin{cases} 2^v & \text{if } \ x = 0 \\ 0 & \text{if } \ x \neq 0 \end{cases}, \tag{2.9}$$

since the left side equals

$$\sum_{z \subseteq x} \langle x, z \rangle \, 2^{v - |x|} = 2^{v - |x|} \sum_{z \subseteq x} (-1)^{|z|}.$$

Using $x + x = 0$ and (2.4) it follows

$$FF^T = 2^v I, \qquad F^{-1} = \frac{1}{2^v} F, \tag{2.10}$$

and we have the inversion formula

$$c(x) = \frac{1}{2^v} \sum_y (-1)^{|xy|} \, \hat{c}(y). \tag{2.11}$$

If we introduce the scalar product

$$(c, d) = \sum_x c(x)\, d(x)\, (= d^T c),$$

we can express (2.10) as invariance

$$(Fc, F^T d) = 2^v (c, d).$$

2.3. Let D_a be the diagonal matrix with $a^{|x|}$ in the (x, x)-position. We can prove directly or apply Section 3 to obtain

$$GD_{-1}G = D_{-1}\,, \qquad HD_{-1}H = D_{-1}\,.$$

Hence we have the inversion formulas

$$\begin{gathered} G^{-1} = D_{-1}GD_{-1}\,, \qquad H^{-1} = D_{-1}HD_{-1}\,, \\ c(x) = (-1)^{|x|} \sum_{y \supseteq x} (-1)^{|y|}\, \hat{c}(y) = (-1)^{x} \sum_{y \subseteq x} (-1)^{|y|}\, \check{c}(y). \end{gathered} \tag{2.12}$$

In terms of the scalar product

$$(c, d)^- = \sum_x (-1)^{|x|}\, c(x)\, d(x) = d^T D_{-1} c,$$

we have the following invariance

$$(Gc, Hd)^- = (c, d)^- = (Hc, Gd)^-.$$

2.4. If we use pointwise multiplication for functions in Q^X we obtain the algebra $Q_\times{}^X$. From (2.6), (2.10), and (2.12) we read off the following result.

THEOREM. *The Fourier transforms F, G, H are algebra-isomorphisms from $Q_*{}^X$, $Q_\wedge{}^X$, $Q_\vee{}^X$, respectively, onto $Q_\times{}^X$. K is simultaneously an algebra-isomorphism from $Q_\wedge{}^X$ onto $Q_\vee{}^X$ and vice versa, and an algebra-automorphism of $Q_\times{}^X$.*

Since $Q_\times{}^X$ is (isomorphic to) the direct product algebra

$$Q \times \cdots \times Q = Q^{2^v},$$

the structure of $Q_*{}^X$, $Q_\wedge{}^X$, $Q_\vee{}^X$ is completely determined.

2.5. At this point it is worth mentioning that one also knows the structure of subalgebras A in $Q_\times{}^X$: For every A there is a partitioning of X into disjoint subsets $Y_0, Y_1, \ldots, Y_m$ ($m \geqslant 0$), where Y_0 may be empty but not the other Y_j ($j \geqslant 1$) and $\bigcup Y_j = X$, such that A consists of exactly those functions $c \in \underset{\sim}{Q}_{\times}{}^X$ which are 0 on Y_0 and constant on each Y_j ($j \geqslant 1$). Furthermore, the partitioning is unique except for the order of the Y_j's ($j \geqslant 1$), and every partitioning is possible. Thus we have the following result: There is a 1-1-correspondence between subalgebras of $\underset{\sim}{Q}_{\times}{}^X$ and these partitionings of X; thereby every subalgebra is isomorphic to $\underset{\sim}{Q}^m$ for some m ($0 \leqslant m \leqslant 2^v$). By Paragraph 2.4 all subalgebras of $\underset{\sim}{Q}_*{}^X, \underset{\sim}{Q}_\wedge{}^X, \underset{\sim}{Q}_\vee{}^X$ can be determined as well. (Note that all of our algebras are commutative.)

3. Combinatorial Matrices

Let $v \geqslant 1$ and consider the 2×2 real matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Using its entries we define the associated combinatorial matrix $M = M(v)$ to be the matrix indexed by the elements of X and having entries:

$$M_{xy} = a^{|V+(x\cup y)|} b^{|y+xy|} c^{|x+xy|} d^{|xy|},$$

where we again use the convention $0^0 = 1$. Observe that the exponents are the cardinalities of the cells of the natural partition of V associated with the subsets x and y. Notice that F, G, H, K, and D_a are all combinatorial matrices derived from $\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right]$, respectively.

We shall prove that all combinatorial matrices have a Kronecker product factorization. Recall that the Kronecker product is defined by

$$A \times B = [a_{ij}B].$$

Also recall that one of its more useful properties is

$$(A_1 \times \cdots \times A_n)(B_1 \times \cdots \times B_n) = A_1B_1 \times \cdots \times A_nB_n .$$

Since every partition of V yields a direct product decomposition of the (semi-) group(s) X_+, $X_\cap$, $X_\cup$, it is clear *a priori* that the related Fourier transforms will be given by matrices which have similar Kronecker product decompositions.

3.1. There is a natural ordering of the finite subsets of $\{1, 2, \ldots\}$:

$$\phi, (1); \quad (2), (2, 1); \quad (3), (3, 1), (3, 2), (3, 2, 1); \quad (4), (4, 1) \ldots .$$

This ordering is defined by identifying each subset $x \subseteq \{1, 2, \ldots\}$ with the $|x|$-tuple $(a_1, a_2, \ldots, a_{|x|})$ where $a_i \in x$ and $a_1 > a_2 > \cdots > a_{|x|}$ and then ordering the tuples lexicographically. A useful property of this ordering is that the elements of X, i.e., the 2^v subsets of the section $\{1, 2,..., v\}$, are the first 2^v elements listed in this ordering of $\{1, 2,...\}$. Thus if Y is the collection of subsets of $\{1, 2,..., v - 1\}$, the induced ordering of X may be described in terms of the induced ordering of Y by $X = Y; (v, Y)$: we start inductively with ϕ, (1); adding on both of these, in order, with 2 adjoined yields ϕ, (1); (2), (2, 1); then adding on all four subsets in order with 3 adjoined; etc.

3.2 Theorem. *Let $v \geqslant 1$. If $M(v)$ is the combinatorial matrix associated with $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and if X has the natural order then $M(v)$ is the Kronecker product of v identical factors:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \cdots \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

One may easily check that $M(1) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ when X is ordered: ϕ, (1). Now assume that $v > 1$; we may partition $X = Y; (v, Y)$ where Y is the collection of all subsets of $W = \{1, 2,..., v - 1\}$. This yields the decomposition

$$M(v) = \left[\begin{array}{c|c} aM(v-1) & bM(v-1) \\ \hline cM(v-1) & cM(v-1) \end{array}\right] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times M(v-1).$$

We may verify this by considering each block separately. For the upper left hand block, let $x, y \in Y$, we have

$$\begin{aligned} M_{xy}(v) &= (a^{|V+(x\cup y)|}b^{|y+xy|}c^{|x+xy|}d^{|xy|}), \\ &= a(a^{|W+(x\cup y)|}b^{|y+xy|}c^{|x+xy|}d^{|xy|}), \\ &= aM_{xy}(v-1). \end{aligned}$$

For the upper right, let $x, y \in Y$ and let $y' = y + \{v\}$, then

$$\begin{aligned} M_{xy'}(v) &= (a^{|V+(x\cup y')|}b^{|y'+xy'|}c^{|x+xy'|}d^{|xy'|}), \\ &= b(a^{|W+(x\cup y)|}b^{|y+xy|}c^{|x+xy|}d^{|xy|}), \\ &= bM_{xy}(v-1). \end{aligned}$$

Similarly for the lower left block. Finally for the lower right block let $x, y \in Y$ and $x' = x + \{v\}$, $y' = y + \{v\}$. We have

$$\begin{aligned} M_{x'y'}(v) &= a^{|V+(x'\cup y')|}b^{|y'+x'y'|}c^{|x'+x'y'|}d^{|x'y'|}), \\ &= d(a^{|W+(x\cup y)|}b^{|y+xy|}c^{|x+xy|}d^{|xy|}), \\ &= dM_{xy}(v-1). \end{aligned}$$

3.3. We may now verify identities among combinatorial matrices by simply computing out the case $v = 1$. In this way we may verify the identities which we have already stated:

$$FF = 2^v I, \qquad GD_{-1}G = D_{-1}, \qquad HD_{-1}H = D_{-1}.$$

Furthermore,

$$F = HD_{-2}G, \qquad G = D_{\frac{1}{2}}HD_{-1}F.$$

Explicitly,

$$\bar{c}(x) = \sum_{y \subseteq x} (-2)^{|y|}\, \hat{c}(y), \tag{3.1}$$

$$\hat{c}(x) = \frac{1}{2^{|x|}} \sum_{y \subseteq x} (-1)^{|y|}\, \bar{c}(x). \tag{3.2}$$

We also have

$$KH = HD_{-1}G, \qquad KG = (-1)^v\, GD_{-1}H; \tag{3.3}$$

from which it follows that

$$\check{c}(x + V) = \sum_{y \subseteq x} (-1)^{|y|}\, \hat{c}(y),$$

$$\hat{c}(x + V) = (-1)^v \sum_{y \supseteq x} (-1)^{|y|}\, \check{c}(y).$$

3.4. We have of course that $\widehat{c \wedge d} = \hat{c}\hat{d}$. We now prove the additional identities

$$\widehat{c * d}(u) = \sum_{x \cup y = u} (-2)^{|xy|}\, \hat{c}(x)\, \hat{d}(y), \tag{3.4}$$

$$\widehat{c \vee d}(u) = \sum_{x \cup y = u} (-1)^{|xy|}\, \hat{c}(x)\, \hat{d}(y). \tag{3.5}$$

Proof of (3.4). We have $F(c * d) = FcFd$; thus,

$$H^{-1}F(c * d) = (H^{-1}Fc) \vee (H^{-1}Fd).$$

However, $H^{-1}F = D_{-2}G$; hence,

$$D_{-2}\widehat{(c * d)} = (D_{-2}\hat{c}) \vee (D_{-2}\hat{d}).$$

To prove (3.5), we start with $H(c \vee d) = (Hc)\,(Hd)$; thus,

$$G^{-1}H(c \vee d) = (G^{-1}Hc) \wedge (G^{-1}Hd).$$

It follows from (3.3) and (2.7) that $G^{-1}H = KD_{-1}G$ and we get

$$KD_{-1}G(c \vee d) = (KD_{-1}Gc) \wedge (KD_{-1}Gd).$$

Finally, by (2.8), we have

$$D_{-1}\widehat{c \vee d} = (D_{-1}\hat{c}) \vee (D_{-1}\hat{d}).$$

These formulas may be used to verify and extend the formulae given in Paragraphs 1.3 and 1.5.

3.5. In [6], fundamental matrices E_i were defined by the v-fold product

$$E_i = I \times \cdots \times I \times E \times I \times \cdots \times I,$$

where $I = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, $E = \left[\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right]$, and E occurs in the i-th position. Since $H(1) = I + E$, we have $H(v) = (I + E) \times \cdots \times (I + E)$. Distributing, we get

$$H(v) = \sum_{x \in X} \prod_{i \in x} E_i .$$

4. t-Exact Covers

According to (1.1) and (2.2) we can describe t-exact covers as those functions $c \in Q^X$ for which

$$\hat{c} = \lambda_s(c) \quad \text{on} \quad V_s \qquad (s = 0, \dots, t).$$

We assume $0 \leqslant t \leqslant v$, of course. Let $Q_t(X)$ denote the set of all functions in Q^X which are constant on each V_s ($s = 0, \dots, t$). Using G^{-1} we see that c is a t-exact cover if and only if

$$\hat{c} \in Q_t(X). \tag{4.1}$$

Now, our results on Fourier transforms will be sufficient to determine the structure of $C_t(X)$.

4.1. Clearly, $Q_t(X)$ is a subalgebra of $Q_\times{}^X$ which is isomorphic to $Q^{m(t)}$, where the dimension $m(t)$ is given by

$$m(t) = (t + 1) + 2^v - \sum_{s=0}^{t} \binom{v}{s} = 2^v + t - \sum_{s=1}^{t} \binom{v}{s}. \tag{4.2}$$

Using the result in Paragraph 2.4 we obtain the following result.

THEOREM. *The Fourier transform G is an isomorphism from $C_t(X)$ onto $Q_t(X)$, in particular $C_t(X)$ is a subalgebra in $Q_\wedge^X$ of dimension $m(t)$.*

Since G is an algebra-isomorphism this determines the structure of $C_t(X)$ within $Q_\wedge^X$.

4.2. The transforms H, FG^{-1}, and GF^{-1} map $Q_t(X)$ into itself as can be seen from (2.3), (3.1), and (3.2). In this context it is useful to think of the members of $Q_t(X)$ as functions in Q^X which depends only on $|x|$ whenever $|x| \leqslant t$. Thus (4.1) is equivalent to $\bar{c} \in Q_t(X)$ which, in turn, means

$$c \in C_t \quad \text{iff} \quad c \in Q^X \qquad \text{and} \qquad \bar{c} = \mu_s(c) \quad \text{on} \quad V_s\,, \qquad s = 0,\dots, t.$$

The formulas (3.1) and (3.2) actually interrelate the parameters $\lambda_0, \dots, \lambda_t$ and the new parameters $\mu_0, \dots, \mu_t$; e.g.,

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(\mu_0 - \mu_1), & \lambda_2 &= \tfrac{1}{4}(\mu_0 - 2\mu_1 + \mu_2) \\ \mu_1 &= \lambda_0 - 2\lambda_1\,, & \mu_2 &= \lambda_0 - 4\lambda_1 + 4\lambda_2\,. \end{aligned}$$

These interrelations have been used in [2] to prove a $\bar{c}$ analog to a fundamental theorem of design theory and led to a characterization of symmetric block designs [3].

Now let $Q^t(X)$ denote the set of all functions in Q^X which are constant on each V_s ($v - t \leqslant s \leqslant v$). Clearly, G maps $Q^t(X)$ into itself while K maps $Q_t(X)$ into $Q^t(X)$ and $Q^t(X)$ into $Q_t(X)$. The representations for HG^{-1} and GH^{-1} imply that $\hat{c} \in Q_t(X)$ is equivalent to $\check{c} \in Q^t(X)$. We gather all of the results together.

THEOREM. *The Fourier transform F is an isomorphism between $C_t(X)$ and $Q_t(X)$ while H is an isomorphism between $C_t(X)$ and $Q^t(X)$. $C_t(X)$ is then a subalgebra of both Q_*^X and $Q_\vee^X$; furthermore, K is an algebra isomorphism between $Q_\wedge^X$ and $Q_\vee^X$ which leaves $C_t(X)$ invariant.*

4.3. The support of c consists of all x with $c(x) \neq 0$; this collection will be denoted by supp (c). Suppose that Y is a nonempty subset of X and denote the collection of all t-exact covers c with supp $(c) \subseteq Y$ by $C_t(Y)$. Clearly $C_t(Y) = C_t(X) \cap Q^Y$, where Q^Y is embedded into Q^X as coordinate subspace. While $Y = V_k$ does not lead to an algebra structure there are other spaces that do.

THEOREM. *If Y is an additive subgroup of X then $C_t(Y)$ is a subalgebra in Q_*^X; if Y is a subsemigroup of X under $\wedge$ or $\vee$ then $C_t(Y)$ is a subalgebra in $Q_\wedge^X$ or $Q_\vee^X$.*

This follows from Paragraphs 4.1 and 4.2 since Q^Y is a subalgebra in $Q_*{}^X$, respectively, $Q_\wedge{}^X$ or $Q_\vee{}^X$, under the conditions mentioned.

By the corresponding Fourier transform $C_t(Y)$ is isomorphic to some subalgebra of Q^X which are known according to Paragraph 2.5. This determines the structure of $C_t(Y)$ within $Q_*{}^X$, respectively, $Q_\wedge{}^X$ or $Q_\vee{}^X$.

We shall discuss two special cases.

4.4. Let Y be the additive subgroup of X consisting of all subsets with even cardinality. Thus $c \in Q^Y$ means $c \in Q^X$ and $c(x) = 0$ wherever $|x|$ odd. By (2.1) and (2.11) it follows that the latter condition is equivalent to

$$\bar{c}(x + V) = \bar{c}(x), \qquad x \in X.$$

Thus $F(Q^Y)$ consists exactly of all functions in Q^X which are constant on the pairs $\{x, x + V\}$. Since

$$F(C_t(Y)) = Q_t(X) \cap F(Q^Y),$$

$F(C_t)$ consists of all functions in Q^X which are constant on

$$V_0 \cup V_v, V_1 \cup V_{v-1}, \ldots, V_t \cup V_{v-t}$$

and on all pairs $\{x, x + V\}$ not mentioned yet. The total number of different constancy-sets is

$$m = \begin{cases} 2^{v-1} + t - \sum_{s=1}^{t} \binom{v}{s}, & \text{if } 0 \leqslant t < \dfrac{v}{2} \\[2ex] \left[\dfrac{v}{2}\right] + 1, & \text{if } \dfrac{v}{2} \leqslant t \leqslant v \end{cases}.$$

Hence the structure within $Q_*{}^X$ is described by

$$C_t(Y) \cong Q^m.$$

If e_k is defined by

$$e_k(x) = \begin{cases} 1 & \text{if } |x| = k \\ 0 & \text{otherwise} \end{cases},$$

then $e_{2k} \in C_t(Y)$ for $0 \leqslant k \leqslant v/2$ and all $0 \leqslant t \leqslant v$. These elements must generate $C_t(Y)$ if $v/2 \leqslant t \leqslant v$.

4.5. Let Y be the subsemigroup of X under $\cap$ consisting of all subsets with cardinality $\leqslant k$, $(t \leqslant k \leqslant v)$. Thus $c \in Q^Y$ means $c \in Q^X$ and $c(x) = 0$ whenever $|x| > k$. By (2.2) and (2.12) it follows that

$$G(Q^Y) = Q^Y;$$

hence $G(C_t(Y))$ consists of all functions in Q^X which are constant on $V_0, \ldots, V_t$ and zero for $|x| > k$. This subalgebra of Q^X has dimension

$$m = t + 1 + \sum_{s=t+1}^{k} \binom{v}{s};$$

thus the structure within Q^X is described by

$$C_t(Y) \cong Q^m.$$

If Y is the subsemigroup of X under $\cup$ consisting of all subsets with cardinality $\geqslant v - k$ $(t \leqslant k \leqslant v)$, then

$$H(Q^Y) = Q^Y, \qquad H(C_t(Y)) = Q^t(X) \cap Q^Y \cong Q^m,$$

where

$$m = t + 1 + \sum_{s=v-k}^{v-t-1} \binom{v}{s} = t + 1 + \sum_{s=t+1}^{k} \binom{v}{s}.$$

It is clear that the algebras $C_t(Y)$, discussed in this section, correspond to each other under K.

5. Reduction Theorems

In this section we relate C_t to C_1 by introducing new spaces $X_t = X(V_t)$ consisting of all subsets of V_t $(0 \leqslant t \leqslant v)$. We define three mappings $\langle t \rangle$, $[t]$, and (t) from $X(V)$ into $X(V_t)$ by:

$$\begin{aligned} x &\to x^{\langle t \rangle} = \{z \mid z \in V_t, \langle z, x \rangle = -1\}, \\ x &\to x^{[t]} = \{z \mid z \in V_t, [z, x] = 1\}, \\ x &\to x^{(t)} = \{z \mid z \in V^t, [x, z] = 0\}. \end{aligned}$$

They will be fundamental in the following discussion.

5.1. It is easy to see that for $x, y \in X$,

$$(x + y)^{\langle t \rangle} = x^{\langle t \rangle} + y^{\langle t \rangle};$$
$$(xy)^{[t]} = x^{[t]} y^{[t]}; \qquad (x \cup y)^{(t)} = x^{(t)} \cup y^{(t)}.$$

We simply check the t-sets which occur as members of each set. There is a natural way to embed X in Q^X:

$$x \to \delta_x, \qquad \text{where} \qquad \delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}.$$

The (semi)group operations are preserved in the following sense:

$$\delta_{x+y} = \delta_x * \delta_y\,, \qquad \delta_{xy} = \delta_x \wedge \delta_y\,, \qquad \delta_{x \cup y} = \delta_x \vee \delta_y\,.$$

From these definitions it follows that $\langle t \rangle$, $[t]$, and (t) extend linearly to algebra homomorphism from $Q_*{}^X$ into $Q_*^{X_t}$, respectively, from $Q_\wedge{}^X$ into $Q_\wedge^{X_t}$, and $Q_\vee{}^X$ into $Q_\vee^{X_t}$. We also have the explicit formulas:

$$c^{\langle t \rangle}(w) = \sum_x c(x), \qquad (x \in X,\ x^{\langle t \rangle} = w);$$

$$c^{[t]}(w) = \sum_x c(x), \qquad (x \in X,\ x^{[t]} = w);$$

$$c^{(t)}(w) = \sum_x c(x), \qquad (x \in X,\ x^{(t)} = w).$$

5.2. On Q^{X_t} we may also define the Fourier Transforms F, G and H by formulae (2.1), (2.2) and (2.3), and we are going to use the same notation as on Q^X. It is not difficult to prove that for $u \in X_t$, $x \in X$,

$$\langle u, x^{\langle t \rangle} \rangle = \left\langle \sum_{z \in u} z, x \right\rangle;$$

$$[u, x^{[t]}] = \left[\bigcup_{z \in u} z, x\right]; \qquad [x^{(t)}, u] = \left[x, \bigcap_{z \notin u} z\right].$$

These equations need only be verified for u consisting of a single t-set in view of (2.4) and (2.5).

We may now compute

$$\begin{aligned} G(c^{[t]})\,(u) &= \sum_w\, [u, w]\, c^{[t]}(w), \\ &= \sum_x\, [u, x^{[t]}]\, c(x), \\ &= G(c) \left(\bigcup_{z \in u} z\right). \end{aligned}$$

Similarly,

$$F(c^{\langle t \rangle})\,(u) = F(c) \left(\sum_{z \in u} z\right),$$

$$H(c^{(t)})\,(u) = H(c) \left(\bigcap_{z \notin u} z\right).$$

5.3. From these results we infer that for $c \in Q^X$,

$$c^{[t]} \in C_1(X_t) \text{ means } \hat{c} \text{ is constant on } V_t\,,$$

$$c^{\langle t\rangle} \in C_1(X_t) \text{ means } \bar{c} \text{ is constant on } V_t\,,$$

and

$$c^{(t)} \in C_1(X_t) \text{ means } \check{c} \text{ is constant on } V_t\,.$$

This yields the following reduction theorem.

THEOREM. *Suppose that* $1 \leqslant t \leqslant v$. *A* $(t-1)$*-exact cover c is t-exact if and only if* $c^{[t]} \in C_1(X_t)$ *or, equivalently,* $c^{\langle t\rangle} \in C_1(X_t)$, *or* $c^{(v-t)} \in C_1(X_{v-t})$. *A function* $c \in Q^X$ *is a t-exact cover if and only if* $c^{[s]} \in C_1(X_s)$ *for* $s = 1,\ldots, t$ *or, equivalently,* $c^{\langle s\rangle} \in C_1(X_s)$ *for* $s = 1,\ldots, t$, *or* $c^{(s)} \in C_1(X_s)$ *for* $s = v - t,\ldots, v$.

6. Integral Covers and Designs

In this section we collect information about t-exact covers with integral multiplicities or with non-negative, integral multiplicities; the latter were called general t-designs in Paragraph 1.6, and we introduced the notations $C_t[X]$ and $C_t^+[X]$ respectively for these collections in Paragraph 1.7. Let Z denote the ring of integers and let N denote the semiring of nonnegative integers. Clearly

$$C_t[X] = C_t(X) \cap Z^X, \qquad C_t^+[X] = C_t(X) \cap N^X. \tag{6.1}$$

6.1. Obviously for every $c \in C_t(X)$ there is a minimal positive integer n such that $c' = nc \in C_t[X]$. This interrelates $C_t(X)$ and $C_t[X]$ most directly:

$$C_t(X) = Q \cdot C_t[X].$$

Let e be defined by $e(x) = 1$ for all x. Clearly $e \in C_t^+(X)$ for $0 \leqslant t \leqslant v$. Furthermore, for every $c \in C_t[X]$ there is a minimal positive integer n such that $c' = c + ne \in C_t^+[X]$. This interrelates $C_t[X]$ and $C_t^+[X]$ most directly:

$$C_t[X] = C_t^+[X] + Z \cdot e.$$

6.2. We observe that Z^X is a ring in $Q_\times{}^X, Q_\wedge{}^X, Q_\vee{}^X, Q_*{}^X$; we employ the notation $Z_\times{}^X$, $Z_\wedge{}^X$, $Z_\vee{}^X$, $Z_*{}^X$ to indicate the multiplication used. Similarly, N^X is a semiring in $Q_\times{}^X, Q_\wedge{}^X, Q_\vee{}^X, Q_*{}^X$ and will be denoted by $N_\times{}^X$, $N_\wedge{}^X$, $N_\vee{}^X$, $N_*{}^X$, respectively. Hence (6.1) in conjunction with Paragraphs 4.1 and 4.2 yields the following result:

THEOREM. *$C_t[X]$ is a ring in $Z_\wedge{}^X$, $Z_\vee{}^X$, and in $Z_*{}^X$; $C_t^+[X]$ is a semiring in $N_\wedge{}^X$, $N_\vee{}^X$, and in $N_*{}^X$.*

6.3. The ring structure of $Z_\wedge{}^X$ and $Z_\vee{}^X$ can be determined since G^{-1} and H^{-1} are both integral. G and H are ring-isomorphisms from $Z_\wedge{}^X$, respectively, $Z_\vee{}^X$ onto $Z_\times{}^X$, which is a direct product of 2^v copies of Z. Let $Z_t(X)$ denote the set of all functions in Z^X which are constant on V_s for $s = 0,..., t$; similarly, let $Z^t(X)$ denote the set of all functions in Z^X which are constant on V_s for $s = v - t,..., v$. Clearly $Z_t(X)$ and $Z^t(X)$ are rings in $Z_\times{}^X$ which are isomorphic to $Z^{m(t)}$ where $m(t)$ is given by (4.2). Hence $C_t[X]$ is characterized by the condition $\hat{c} \in Z_t(X)$ or $\check{c} \in Z^t(X)$; we have proved the following result.

THEOREM. *The Fourier transforms G and H are ring isomorphisms from $C_t[X]$ (as a subring of $Z_\wedge{}^X$ and $Z_\vee{}^X$) onto $Z_t(X)$ and $Z^t(X)$, respectively (as subrings of $Z_\times{}^X$), in particular $C_t[X] \cong Z^{m(t)}$.*

6.4. The Fourier transform F is a ring-isomorphism from $Z_*{}^X$ into $Z_\times{}^X$, hence $C_t[X]$ (as subring of $Z_*{}^X$) will be isomorphic to some subring of $Z_\times{}^X$. But the structure of these has not yet been determined. Thus one should like to determine the structures of $Z_*{}^X$, $N_\wedge{}^X$, $N_\vee{}^X$, and $N_*{}^X$, and of their appropriate sub(semi)rings.

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